UDC 531.31+62-50

ON PROBABILITY CONTROL OF CERTAIN SYSTEMS

PMM Vol. 40, № 5, 1976, pp. 782-789
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(Received September 4, 1975)

The problem of maximizing the probability of a controlled system attaining the specified state is considered. The related Bellman equations and the methods for determining the optimal control are investigated. Examples are presented.

1. The following controlled system is given:

$$x^{\cdot}(t) = f(x(t)) + Bu(x(t)) + \sigma(x(t))\xi^{\cdot}(t), \quad t \geqslant 0 \tag{1.1}$$

where x is the phase coordinate vector from a Euclidean space E_n, B is a constant matrix the function f and the matrix σ satisfy the requirement

$$|f(x_1) - f(x_2)| + |\sigma(x_1) - \sigma(x_2)| \leqslant c\,|x_1 - x_2| \tag{1.2}$$

for some constant $c \geqslant 0$ and for any $x_1, x_2 \in E_n$; finally, $\xi(t)$ is a standard Wiener process and the control $u(x) \in E_m$, where

$$u(x) \in U \tag{1.3}$$

for a given bounded convex set U. We note that a strong solution of Eq. (1.1) exists [1] when conditions (1.2) are satisfied and $u \equiv 0$. Let there be a bounded target set $Q \subset E_n$ with a boundry Q_1. We denote the complement of $Q \cup Q_1$ in E_n by Q_2. Further, let $\tau_x(u)$ denote the instant that the system (1.1) first attains Q_1 under control u and initial condition $x(0) = x \in Q_2$. In this connection $\tau_x(u)$ is assumed to equal infinity for those realizations of process (1.1) which do not attain Q_1 in any finite time. A certain control from (1.3) is said to be admissible if for this control and for the initial condition $x(0) \in Q_2$ a solution of Eq. (1.1) exists. By $P(\cdot)$ we denote the probability of the event within the parentheses.

Problem. Among the admissible controls choose the one which maximizes the probability $P(\tau_x(u) < \infty)$, $x \in Q_2$, i.e., maximizes the probability of attaining the target set Q in finite time, when starting from Q_2.

With the problem posed we connect the following boundary-value problem for the Bellman equation:

$$\begin{aligned} &\max\nolimits_{u \in U} L_u V(x) = 0, \quad x \in Q_2 \\ &V(x) = 1, \quad x \in Q_1 \\ &L_u = (f + u)' \frac{\partial}{\partial x} + \frac{1}{2} \operatorname{Tr} \sigma\sigma' \frac{\partial^2}{\partial x^2} \end{aligned} \tag{1.4}$$

here the prime is the sign for the transpose, the vector $\partial / \partial x$ and the matrix $\partial^2 / \partial x^2$ have, respectively, the components $\partial / \partial x_i$ and $\partial^2 / \partial x_i \partial x_j$, $i, j = 1, \ldots n$, the symbol Tr denotes the trace of the matrix. In the usual manner [2] we can establish that if the function

$$\max_{u\in U} P(\tau_x(u) < \infty), \quad x \in Q_2 \tag{1.5}$$

is sufficiently smooth in x, it satisfies (1.4). However, even then the relations (1.4), in general, do not determine the function (1.5) because of the possible non-uniqueness of the solution of the exterior boundary-value problem (1.4). In addition, the usual procedure for computing the optimal control, consisting in solving the Bellman equation and subsequently determining the optimal control from it, is also not always realizable in the problem posed because, for example, the situation when the function (1.5) identically equals unity is possible. In Sects. 2 and 3 we shall given certain answers to the questions raised, while in Sect. 4 we give examples. We note that other problems with a probabilistic criterion were investigated in [3,4]. To be precise, the problem of maximizing the probability of the system staying outside a prescribed region was considered in [3] and the problem of maximizing the probability of staying within a region during a specified finite time interval was considered in [4].

2. In what follows we assume that the matrix $\sigma(x)$ is uniformly nonsingular with respect to $x \in Q_2$ and that Q_1 is a Liapunov type boundary [5]. Let $R_N \subset E_n$ be a sequence of spheres of radius N with boundaries r_N, where $Q \subset R_N$, and let the sequence of functions $V_N(x)$ be defined by the relations

$$\begin{aligned} &\max_{u\in U} L_u V_N(x) = 0, \quad x \in Q_2 \cap R_N \\ &V_N(x) = 1, \quad x \in Q_1, \ V_N(x) = 0, \ x \in r_N \end{aligned} \tag{2.1}$$

On the basis of [6] the solution of boundary-value problem (2.1) exists and, moreover, is unique, while by virtue of the maximum principle

$$1 > V_{N_2}(x) \geqslant V_{N_1}(x) > 0, \quad x \in Q_2 \cap R_{N_1}, \quad N_2 \geqslant N_1 \tag{2.2}$$

for elliptic equations [7].

Theorem 1. Let the assumptions made regarding the coefficients of system (1.1) and the region Q be satisfied. Then $V_N(x)$ converges uniformly to $V_0(x)$ as $N \to \infty$ on any bounded set of variation of argument x, while the limit $V_0(x)$ is the minimal positive solution of problem (1.4).

Proof. Let there be given some bounded region $Q_3 \subset E_n$ with a Liapunov type boundary. From the theorem's hypotheses, estimate (2.2) and [6] it follows that

$$\max_x \left\| \frac{\partial V_N(x)}{\partial x} \right\| \leqslant c_1, \quad x \in Q_3 \cap Q_2 \tag{2.3}$$

Similarly, from (2.2), (2.3), (1.2) and [6]

$$\max_x \left\| \frac{\partial^2 V_N(x)}{\partial x^2} \right\| \leqslant c_2, \quad x \in Q_3 \cap Q_2 \tag{2.4}$$

In relations (2.3), (2.4) and below c_i denote certain nonnegative constants. From (2.3), (2.4) and [8] follows the compactness of the sequence $(V_N(x), \partial V_N(x) / \partial x)$ in the space of continuous functions for $x \in Q_3 \cap Q_2$. We choose some convergent subsequence of the sequence $(\partial V_N(x) / \partial x, \ V_N(x))$ and we denote the corresponding limit by $(\partial V_0(x) / \partial x, \ V_0(x))$. Because of the arbitrariness of Q_3 the function $V_0(x)$ is a solution of boundary-value problem (1.4) (see [7]). Hence, from (2.2) and (2.3) it follows that the sequence $V_N(x)$ itself converges uniformly, not decreasing to $V_0(x)$.

We now assume that $V_0(x)$ is not a minimal positive solution of problem (1.4). Then a solution $V(x)$ of this problem exists such that

$$0 \leqslant V(x) \leqslant V_0(x) \tag{2.5}$$

Thus, in particular, the formulas $V(x) = V_N(x) = 1, \quad x \in Q_1$

$$V(x) \geqslant V_N(x) = 0, \quad x \in r_N$$

are valid. Hence from the maximum principle for elliptic equations (see [7]) there follows the estimate

$$V_N(x) \leqslant V(x), \quad x \in Q_2 \cap R_N \tag{2.6}$$

From this estimate, the convergence of $V_N(x)$ to $V_0(x)$ and (2.5) we conclude that $V(x) \equiv V_0(x)$. Theorem 1 is proved.

Note. For the case of an uncontrolled Wiener process (i.e., when in (1.1) the coefficients $f = 0$ and $B = 0$ and σ is that unit matrix) Theorem 1 turns into an assertion established earlier in [9].

In accordance with [10] a measurable function $u_N(x)$ realizing the maximum in (2.1) exists. We introduce the sequence of controls $v_N(x)$ equal to $u_N(x)$ when $x \in Q_2 \cap R_N$ and equal to an arbitrary fixed constant from U for the remaining values of x.

Theorem 2. Under the hypotheses of Theorem 1 the quantity

$$J(x, u) = P(\tau_x(u) < \infty) \leqslant V_0(x) \tag{2.7}$$

for any admissible control u. Further, if the controls $u_N(x)$ are admissible, then as $N \to \infty$

$$J(x, v_N) \to V_0(x) \tag{2.8}$$

where the convergence is uniform on any bounded set of variation of x

Proof. Assume that

$$J(x, u) > V_0(x) \tag{2.9}$$

for some admissible control u. Let $x(t, u)$ denote a solution of (1.1) under control u and let $\tau_x^N(u)$ be the instant of first exit of process $x(t, u)$ from $Q_2 \cap R_N$ under the initial condition $x(0, u) = x \in Q_2 \cap R_N$. On the basis of [11] and of the hypotheses of Theorem 2

$$M\tau_x^N(u) < \infty$$

From here it follows that (see [12])

$$MV_N(x(\tau_x^N(u), u)) - V_N(x) = M \int_0^{\tau_x^N(u)} L_u V_N(x(t, u))\, dt \leqslant$$

$$M \int_0^{\tau_x^N(u)} \max_u L_u V_N(x(t, u))\, dt = 0$$

Hence with due regard to the boundary condition for V_N, we have that

$$P(x(\tau_x^N(u), u) \in Q_1) \leqslant V_N(x) \tag{2.10}$$

for any N. The left-hand side of this inequality tends to $J(x, u)$ as $N \to \infty$. Therefore, in view of (2.5), (2.6), (2.10) and Theorem 1 we conclude that $J(x, u) \leqslant V_0(x)$ which contradicts (2.9). By the same token relation (2.7) has been established.

To prove (2.8) we note that when $x \in Q_2 \cap R_N$ the function $J(x, v_N)$ satisfies the boundary-value problem

$$L_{u_N} J(x, v_N) = 0, \quad x \in Q_2 \cap R_N$$
$$J(x, v_N) = 1, \quad x \in Q_2, \quad J(x, v_N) \geqslant 0, \quad x \in r_N$$

Hence, from (2.1), (2.7) and the maximum principle [7] it follows that

$$V_N(x) \leqslant J(x, v_N) \leqslant V_0(x)$$

The validity of the uniform convergance in (2.8) is also confirmed by Theorem 1. Theorem 2 has been proved. It shows that if the control $v_N(x)$ is used in system (1.1), the functional (1.5) can be made to approximate the optimal value $V_0(x)$ as closely as required if N is sufficiently large.

Note. We note that the method used in the proof of Theorems 1 and 2, consisting of constructing a sequence $V_N(x)$ and passing to the limit as $N \to \infty$, can also serve for a practical computation of the minimal positive solution of the Bellman equation (see Example 1).

3. Let us consider certain properties of function $V_0(x)$. As we have already noted, if $V_0(x) \equiv 1$, then (1.4) does not define the optimal control. By analogy with [13] a sufficient condition for the identity $V_0(x) \equiv 1$ is the existance of a unique solution of boundary-value problem (1.4) in the class of bounded functions, since a bounded function $V_0(x)$ satisfies (1.4) in view of Theorem 1, while the function $V(x) \equiv 1$ also is a solution of (1.4). Another sufficient condition for the identity $V_0(x) \equiv 1$ is the existance, for some admissible control u of a nonnegative function $W(x)$ such that $L_u W(x) \leqslant -c$ for some constant $c > 0$. This last condition follows from [14].

We cite another case when $V_0(x)$ does not define the optimal control. We assume that for some $N > 0$

$$\min_{x,\ |x| \geqslant N} V_0(x) > 0 \tag{3.1}$$

Without loss of generality we can assume

$$V_0(x) < 1, \quad x \in Q_2 \tag{3.2}$$

because otherwise $V_0(x) \equiv 1$ on the basis of the maximum principle for elliptic equations; this situation has already been considered above. Let u_0 be any admissible control maximizing the expression $L_u V_0(x)$. We show that it is not optimal. We have (see [9])

$$MV_0(x(\tau_x^N(u_0), u_0)) - V_0(x) = 0 \tag{3.3}$$

The probability $P(\tau_x^N(u_0) < \infty) = 1$ for any N; therefore, with due regard to inequality (3.2) and the results in Sect. 2 we obtain

$$P(x(\tau_x^N(u_0), u_0) \in r_N) \geqslant c_3 > 0$$

Hence from (3.1) and (3.3) follows the estimate for some $\delta > 0$

$$P(x(\tau_x^N(u_0), u_0) \in Q_1) \leqslant V_0(x) - \delta$$

then, passing to the limit as $N \to \infty$ we conclude that

$$J(x, u_0) = P(x(\tau_x(u_0), u_0) \in Q_1) < V_0(x) \tag{3.4}$$

If now the controls u_N are admissible, then in view of (2.8) relation (3.4) shows that the control u_0 is not optimal. In the contrary case, to prove the above we proceed as follows. We fix an arbitrary point $x_0 \in Q_2$ and a number $\varepsilon > 0$ for which

$$J(x_0, u_0) = V_0(x_0) - \varepsilon \tag{3.5}$$

Using Theorem 1, we take N such that

$$V_0(x_0) - V_N(x_0) \leqslant \varepsilon / 4 \tag{3.6}$$

Further, from the proof of Lemmas 2.1 and 2.2 from [2] follows the existence of an admissible control v such that

$$v'B' \frac{\partial V_0(x)}{\partial x} \geqslant \max_{u \in U} u'B' \frac{\partial V_0(x)}{\partial x} - \varepsilon_1$$

for any $\varepsilon_1 > 0$

We now apply Dynkin's formula [9] to function $V_N(x)$ and to process $x(t, v)$. We have

$$MV_N(x(\tau_{x_0}{}^N(v), v)) - V_N(x_0) = M \int_0^{\tau_{x_0}{}^N(v)} L_v V_N(x(t, v))\, dt \geqslant$$

$$\geqslant M \int_0^{\tau_{x_0}{}^N(v)} (\max_{u \in U} L_u V_N(x(t, v)) - \varepsilon_1)\, dt = -\varepsilon_1 M \tau_{x_0}{}^N(v) \tag{3.7}$$

Hence it follows that

$$P(x(\tau_{x_0}{}^N(v), v) \in Q_1) \geqslant V_N(x_0) - \varepsilon_1 M \tau_{x_0}{}^N(v) \tag{3.8}$$

But in view of the nonsingularity of matrix $\sigma(x)$ and of the boundedness of set U, the quantity of $M\tau_{x_0}{}^N(v)$ is uniformly bounded with respect to $v \in U$ for any admissible control v from (1.3). Therefore, we can choose ε_1 such that

$$P(x(\tau_{x_0}{}^N(v), v) \in Q_1) \geqslant V_N(x_0) - \varepsilon / 4$$

If we now construct control v_0 according to Theorem 2, i.e., set $v_0(x) = v(x)$, $x \in Q_2 \cap R_N$, and take $v_0(x)$ to be a constant from U for the remaining values of x then

$$J(x_0, v_0) \geqslant V_N(x_0) - \varepsilon / 4$$

Hence from (3.5), (2.2) and Theorem 2.1 follows the estimate

$$J(x_0, v_0) \geqslant V_0(x_0) - \varepsilon / 2$$

From this estimate and (3.5) follows that the control $u_0(x)$ under condition (3.1) is not optimal. Let us consider the case when

$$\lim_{|x| \to \infty} V_0(x) = 0 \tag{3.9}$$

We show that when equality (3.9) is satisfied, any admissible control u_0 maximizing the expression $L_u V_0$ is optimal. For this we make use of formula (3.3) and we present the first term in the form

$$MV_0(x(\tau_x^N(u_0),u_0)) = P(x(\tau_x^N(u_0),\ u_0) \in Q_1) + M_1V_0(x(\tau_x^N(u_0),u_0)) \quad (3.10)$$

where the symbol M_1 implies that the mean is computed over trajectories for which $x(\tau_x^N(u_0),\ u_0) \in r_N$. In (3.10) we pass to the limit as $N \to \infty$. With due regard to (3.3), (3.9) and (3.10) we obtain $J(x,\ u_0) = V_0(x)$. By the same token we have established the optimality of control u_0 on the basis of Theorem 2.

We show that a sufficient condition for (3.9) is the existence of a positive continuous function $W(x)$ for which

$$\max_{u\in U} L_u W(x) \leqslant 0, \quad x \in Q_2, \quad \lim_{|x|\to\infty} W(x) = 0 \quad (3.11)$$

As in the derivation of formulas (3.7) and (3.8), for any $\varepsilon > 0$ we can find an admissible control v such that for a given N

$$P(x(\tau_{x_0}^N(v),\ v) \in Q_1 \geqslant V_N(x_0) - \varepsilon \quad (3.12)$$

where x_0 is an arbitrary fixed point from $Q_2 \cap R_N$. Further, by replacing the function V_N in (3.7) by W, with due regard to (3.11) we have

$$MW(x(\tau_{x_0}^N(v), v)) - W(x_0) \leqslant M \int_0^{\tau_{x_0}^N(v)} \max_{u\in U} L_u W(x(t,v))dt \leqslant 0$$

Hence, from the positiveness of W and from (3.10) it follows that

$$c_4W(x_0) > P(x(\tau_{x_0}^N(v), v) \in Q_1) \quad (3.13)$$

Because the point x_0 and the number ε are arbitrary, estimates (3.12) and (3.13) imply that $c_4W(x) \geqslant V_N(x)$. Hence $c_4W(x) \geqslant V_0(x)$. Equation (3.9) follows from the latter inequality and (3.11).

Note. The question of the optimality of the control $u_0(x)$ maximizing the quantity $L_u V_0(x)$ reduces to the question of the existence of a solution of system (1.1) with $u = u_0(x)$. In the general case control $u_0(x)$ will be only a measurable function [10], which is insufficient for the existance of a solution of system (1.1) in the strong sense. Therefore, in that situation it is necessary to extend the concept of a solution, understanding it in the weak sense [1] or to impose further constraints on the parameters of the problem, sufficient for a solution in the strong sense to exist when $u = u_0(x)$. Under the assumption of Sect. 2 we formulate certain sufficient conditions for the existence of a solution in the weak and strong sense mentioned when $u = u_0(x)$:

1) let matrix σ in (1.1) be constant. Then (see [1], p.143) a weak solution of system (1.1) exists when $u = u_0(x)$,

2) let constraint (1.2) have the form $\| u \| \leqslant c,\ c > 0$; then $u_0(x)$ is continuous in x and, consequently, system (1.1) has a strong solution when $u = u_0(x)$ (see [1]).

We note further that when investigating actual controlled systems, at first we can formally determine a function $u_0(x)$ maximizing L_uV_0 and next prove the optimality of $u_0(x)$ by using any of the requirements sufficient for the existence of a solution of system (1.1) when $u = u_0(x)$ (see Example 1).

4. Example 1. We consider the motion of a rigid body relative to its centre of mass, assuming that equations of motion (the Euler equations) are

$$x_1^{\cdot} = x_2x_3(a_2 - a_3)(a_2a_3)^{-1} + u_1 + \sigma\xi_1^{\cdot} \quad (1\ 2\ 3)$$
$$|u| = (u_1^2 + u_2^2 + u_3^2)^{1/2} \leqslant b, \quad b > 0$$

Here the x_i are the projections of the vector of the body's moment of momentum relative to the centre of the mass onto the principal central inertial axes, a_i are the principal central inertial moments, the number $\sigma > 0$, the symbol (1 2 3) denotes that the equations for x_2 and x_3 are obtained from (4.1) by a cyclic permutation of the indices. Let Q be a sphere of given radius r. In this case the Bellman Eq. (1.4) does not have a unique solution. In fact, with due regard to (4.1) any function

$$1 + c\int_r^{|x|}\left(\frac{r}{s}\right)^2 \exp\left(\frac{2b}{\sigma^2}(r-s)\right)ds, \quad |x| \geqslant r$$

is a solution of (1.4) for any arbitrary constant $c \geqslant 0$

Using Theorem 2 we get that the optimal control in this problem is $u_0 = -c_1|x|^{-1}x$, where c_1 is an arbitrary number from $[0, b]$, while the probability of attaining Q corresponding to this control, equals unity. To prove the existance of a solution of Eq. (4.1) it is sufficient [14] to find the Liapunov function $W \geqslant 0$ for which $L_{u_0}W \leqslant c_2W$ for some constant c_2. In the present example we can take the function

$$W = \frac{1}{c_1}\left[|x| - r + \frac{\sigma^2}{b}\ln\frac{|x|}{r} + \frac{\sigma^4}{2b^2}\left(\frac{1}{r} - \frac{1}{|x|}\right)\right], \quad |x| > r$$

as the Liapunov function and with its help be convinced of the existence of a solution of (4.1) up to the instant of attaining Q, because $L_{u_0}W(x) = -1$, $|x| > r$. When investigating certain actual systems we can relax the requirement on the nonsingularity of the diffusion matrix σ. We present an appropriate example.

Example 2. We study the precessional motion of a planar gyroscopic pendulum [15]. Let the controlling moment be applied around the gyroscope's housing axis, x_1 be the pendulum's rotation angle around its axis, and x_2 be the housing's rotation angle. Under certain assumptions the equations of motion are [16]

$$x_1^{\cdot} = a_1x_2 + u, \quad |u| \leqslant 1, \quad t \geqslant 0, \qquad x_2^{\cdot} = -a_2x_1 - a_3x_2 + \sigma\xi$$

Here $a_i > 0$ and $\sigma > 0$ are constants whose mechanical meanings have been presented, for instance, in [16]. We pose the problem of bringing (4.2) into the sphere $Q = x_1^2 + x_2^2 \leqslant r^2$.

Let us show that for this problem the optimal control is

$$u_0(x) = -\operatorname{sgn} x_1$$

while the probability of attaining under this control equals unity. By N we denote a number for which $\sigma^2 < 2|x_1|a_2 + 2a_3a_1x_2^2$ follows from the inequality $x_1^2 + x_2^2 \geqslant N$ while by Q_3 we denote the set

$$Q_3 = Q \cup (x_1, x_2 : |x_2| \leqslant \varepsilon, \ |x_1| \leqslant N), \qquad 0 < \varepsilon < \min(r, 1/(2a_1))$$

We assume

$$\omega(x) = \frac{a_2}{a_1} x_1^2 + x_2^2 - \alpha \left(\frac{a_2}{a_1} x_1^2 + x_2^2 \right)^{-m}$$

In view of (4.2) and (4.3) we conclude that we can so choose the constants $\alpha > 0$ and $m > 0$ that

$$L_{u_0} \omega(x) \leqslant -c < 0, \quad x \in \overline{Q}_3 \tag{4.4}$$

where $c > 0$ is some constant and $\overline{Q}_3$ is the complement of Q_3 with respect to E_2. From (4.4) and [14] it follows that Q_3 is attained with probability one for any of the ε indicated above.

Now let $x = (x_1, x_2)$ be an arbitrary point $Q_3 \setminus Q$. We show that the probability of attaining Q when starting from x equals unity. For definiteness let $x_1 < 0$. Then $u_0 = 1$ up to the instant τ_0 at which the trajectory $x_1(t)$ reaches zero, and, therefore, on the basis of (4.2)

$$x_2^{\cdot}(t) = -a_3 x_2 + \sigma\xi + a_2 \left[-x_1 e^{a_1 t} + \frac{1}{a_1} (1 - e^{a_1 t}) \right] \tag{4.5}$$

From this and [17] follows the absolute continuity of the measure of process x_2 relative to the Wiener process. Therefore, the probability of process $x_2(t)$ remaining in the region $|x_2| \leqslant 1/(2a_1)$ in the interval $0 \leqslant t \leqslant 2N$ is positive. Hence, with due regard to (4.2) process x_2 originating at $x \in Q_3 \setminus Q$ is able to reach the sphere Q with positive probability within the time $0 \leqslant t \leqslant 2N$. Moreover, in view of (4.5), process $x_2(t)$ is continuous in accordance with the intitial condition for x up to the instant $\min(\tau_0, 2N)$. Consequently, we finally conclude that the upper bound of the probability of reaching the surface of a sphere of radius $N + \varepsilon$ before reaching Q, when starting from Q_3 is less than unity. This (see [14])proves the validity of statements in the example.

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Translated by N. H. C.